

Homotopy and Crossings of Systems of Curves on a Surface

A. Schrijver

Department of Econometrics

Tilburg University

P.O. Box 90153

5000 LE Tilburg, The Netherlands

and

Mathematical Centre,

Kruislaan 413,

1098 SJ Amsterdam, The Netherlands

Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Robert E. Bixby

ABSTRACT

Let C_1, \dots, C_k and $C'_1, \dots, C'_{k'}$ be closed curves on a compact orientable surface S . We characterize (in terms of counting crossings) when there exists a permutation π of $\{1, \dots, k\}$ such that, for each $i = 1, \dots, k$, $C'_{\pi(i)}$ is freely homotopic to C_i or C_i^{-1} . The characterization is equivalent to the nonsingularity of a certain infinite symmetric matrix.

We prove the following theorem:

THEOREM. *Let C_1, \dots, C_k and $C'_1, \dots, C'_{k'}$ be primitive closed curves on a compact orientable surface S . Then the following are equivalent:*

(i) $k = k'$, and there exists a permutation π of $\{1, \dots, k\}$ such that for each $i = 1, \dots, k$

$$C'_{\pi(i)} \sim C_i \quad \text{or} \quad C'_{\pi(i)} \sim C_i^{-1};$$

(ii) for each closed curve D on S ,

$$\sum_{i=1}^k \text{mincr}(C_i, D) = \sum_{i=1}^{k'} \text{mincr}(C'_i, D). \quad (1)$$

Here we use the following terminology and notation. A *closed curve* C on S is a continuous function $C: S_1 \rightarrow S$, where S_1 is the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Two closed curves C and C' are called (*freely*) *homotopic*, and we write $C \sim C'$, if there exists a continuous function $\Phi: [0, 1] \times S_1 \rightarrow S$ such that $\Phi(0, z) = C(z)$ and $\Phi(1, z) = C'(z)$ for all $z \in S_1$. A closed curve C is *primitive* if there do not exist a closed curve D and an integer $n \geq 2$ such that $C \sim D^n$. For a closed curve D and integer n , D^n is the closed curve defined by $D^n(z) := D(z^n)$ for $z \in S_1$.

Finally, for closed curves C and D :

$$\begin{aligned} \text{cr}(C, D) &:= |\{(y, z) \in S_1 \times S_1 \mid C(y) = D(z)\}| \quad \text{if } C \neq D, \\ \text{cr}(C, C) &:= |\{(y, z) \in S_1 \times S_1 \mid C(y) = C(z), y \neq z\}|, \end{aligned} \quad (2)$$

$$\text{mincr}(C, D) := \min\{\text{cr}(\tilde{C}, \tilde{D}) \mid \tilde{C} \sim C, \tilde{D} \sim D\}.$$

Proof. The implication (i) \Rightarrow (ii) being trivial [as $\text{mincr}(C^{-1}, D) = \text{mincr}(C, D)$], we show (ii) \Rightarrow (i). By symmetry we may assume

$$\sum_{i=1}^{k'} \sum_{j=1}^{k'} \text{mincr}(C'_i, C'_j) \leq \sum_{i=1}^k \sum_{j=1}^k \text{mincr}(C_i, C_j). \quad (3)$$

By a result of Baer [1] there exist $\tilde{C}_1 \sim C'_1, \dots, \tilde{C}_{k'} \sim C'_{k'}$ such that

$$\text{cr}(\tilde{C}_i, \tilde{C}_j) = \text{mincr}(C'_i, C'_j) \quad \text{for } i, j = 1, \dots, k'. \quad (4)$$

We may assume that in fact $\tilde{C}_i = C'_i$ for $i = 1, \dots, k'$, that $C'_i \neq C'_j$ if $i \neq j$, and that each point of S is passed at most twice by the C'_i (so no two crossings of the C'_i coincide).

Let $G = (V, E)$ be the graph made up by the curves C'_i . So G is a graph embedded on S . Each point of S passed twice by the C'_i is a vertex, of degree 4, of G . Moreover, we take as vertices some of the points of S passed exactly once by the C'_i , in such a way that G will be a graph without loops or parallel edges. So each vertex of G has degree 2 or 4.

Now by (1), for each closed curve $D: S_1 \rightarrow S \setminus V$,

$$\sum_{i=1}^k \text{mincr}(C_i, D) = \sum_{i=1}^{k'} \text{mincr}(C'_i, D) \leq \sum_{i=1}^{k'} \text{cr}(C'_i, D) = \text{cr}(G, D), \quad (5)$$

where $\text{cr}(G, D) := |\{z \in S_1 \mid D(z) \in G\}|$. Hence, by the “homotopic circulation theorem” in [2], there exist cycles $C_{11}, \dots, C_{1t_1}, \dots, C_{k1}, \dots, C_{kt_k}$ in G and rationals $\lambda_{11}, \dots, \lambda_{1t_1}, \dots, \lambda_{k1}, \dots, \lambda_{kt_k} > 0$ such that

$$C_{ij} \sim C_i \quad (i = 1, \dots, k; j = 1, \dots, t_i), \tag{6a}$$

$$\sum_{j=1}^{t_i} \lambda_{ij} = 1 \quad (i = 1, \dots, k), \tag{6b}$$

$$\sum_{i=1}^k \sum_{j=1}^{t_i} \lambda_{ij} \chi^{C_{ij}}(e) \leq 1 \quad (e \in E). \tag{6c}$$

Here a *cycle* in G is a sequence

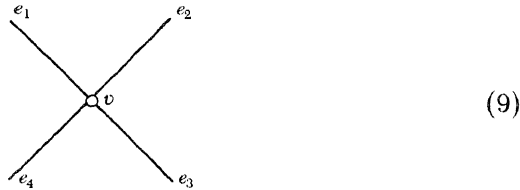
$$C = (v_0, e_1, v_1, e_2, v_2, \dots, v_{d-1}, e_d, v_d), \tag{7}$$

where v_0, \dots, v_d are vertices of G , $v_0 = v_d$, and e_i is the edge connecting v_{i-1} and v_i ($i = 1, \dots, d$). ($v_1, \dots, v_d, e_1, \dots, e_d$ need not be distinct.) With each cycle in G we can associate in the obvious way a closed curve on S —unique up to homotopy. For any cycle (7) and any edge e of G ,

$$\chi^C(e) := \text{number of } i \in \{1, \dots, d\} \text{ with } e_i = e. \tag{8}$$

We call a cycle (7) *nonreturning* if $e_i \neq e_{i-1}$ for $i = 1, \dots, d$, and $e_1 \neq e_d$. Clearly, we may assume the C_{ij} to be nonreturning.

Consider now any vertex v of G of degree 4, and denote the edges incident with v by e_1, e_2, e_3, e_4 in cyclic order:



We call edges e_1 and e_3 *opposite in v*, and similarly, we call e_2 and e_4 *opposite in v*.

The remainder of this proof consists of showing:

- (i) for each edge e , equality holds in (6c);
- (ii) for each cycle C_{ij} and each vertex v of degree 4, each time when C_{ij} passes v , it goes from one edge to the edge opposite in v . (10)

Having shown this, it follows that each C_{ij} belongs to $\{C'_1, (C'_1)^{-1}, \dots, C'_k, (C'_k)^{-1}\}$, and hence we have (i) in our theorem.

In order to prove (10), we first show two lemmas. For each vertex v of degree 4, we fix one choice e_1, e_2, e_3, e_4 as in (9). For any cycle C in G , any vertex of degree 4, in G , and any $i, j \in \{1, 2, 3, 4\}$, let

$$\alpha_{ij}^v(C) := \text{number of times } C \text{ passes } v \text{ by going} \\ \text{from } e_i \text{ to } e_j \text{ or from } e_j \text{ to } e_i. \quad (11)$$

LEMMA A. For any pair of nonreturning cycles C, D in G ,

$$\text{mincr}(C, D) \leq \\ \sum_{v \in W} \left\{ \beta_{13}\gamma_{24} + \beta_{24}\gamma_{13} + \frac{1}{2} [(\beta_{12} + \beta_{34})(\gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24}) \right. \\ \left. + (\beta_{13} + \beta_{24})(\gamma_{12} + \gamma_{14} + \gamma_{23} + \gamma_{34}) + (\beta_{14} + \beta_{23})(\gamma_{12} + \gamma_{13} + \gamma_{24} + \gamma_{34}) \right\}, \quad (12)$$

where $\beta_{ij} := \alpha_{ij}^v(C)$ and $\gamma_{ij} := \alpha_{ij}^v(D)$, and $W := \{v \in V \mid v \text{ has degree } 4\}$.

[Note that the term in (12) with factor $\frac{1}{2}$ contains all products $\beta_{gh}\gamma_{ij}$ with $g \neq h$, $i \neq j$, and $|\{g, h\} \cap \{i, j\}| = 1$.]

Proof of Lemma A. We can represent C and D as

$$C = (v_0, f_1, v_1, f_2, v_2, \dots, v_{s-1}, f_s, v_s), \\ D = (w_0, g_1, w_1, g_2, w_2, \dots, w_{t-1}, g_t, w_t), \quad (13)$$

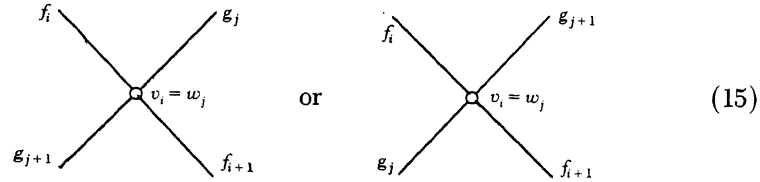
where v_0, \dots, v_s and w_0, \dots, w_t are vertices of G with $v_s = v_0$ and $w_t = w_0$, f_i is an edge of G connecting v_{i-1} and v_i ($i = 1, \dots, s$), and g_i is an edge of

G connecting w_{i-1} and w_i ($i = 1, \dots, t$), so that $f_i \neq f_{i-1}$ for $i = 1, \dots, s$ and $g_i \neq g_{i-1}$ for $i = 1, \dots, t$ (taking indices of v and $f \bmod s$, and indices of w and $g \bmod t$).

Let λ be the number of pairs $(i, j) \in \{1, \dots, s\} \times \{1, \dots, t\}$ such that

$$v_i = w_j \in W, f_i \text{ and } f_{i+1} \text{ are opposite in } v_i, \text{ and } g_j \text{ and } g_{j+1} \text{ are opposite in } w_j, \text{ while } \{f_i, f_{i+1}\} \neq \{g_j, g_{j+1}\}. \quad (14)$$

So (14) corresponds to



Let μ be the number of pairs $(i, j) \in \{1, \dots, s\} \times \{1, \dots, t\}$ such that

$$v_i = w_j \in W, \quad f_{i+1} = g_{j+1}, \quad \text{and} \quad f_i \neq g_j. \quad (16)$$

So μ is also equal to the number of pairs $(i, j) \in \{1, \dots, s\} \times \{1, \dots, t\}$ such that

$$v_i = w_j \in W, \quad f_i = g_j, \quad \text{and} \quad f_{i+1} \neq g_{j+1}. \quad (17)$$

Similarly, let ν be the number of pairs $(i, j) \in \{1, \dots, s\} \times \{1, \dots, t\}$ such that

$$v_i = w_j \in W, \quad f_{i+1} = g_j, \quad \text{and} \quad f_i \neq g_{j+1}. \quad (18)$$

Again, ν is also equal to the number of pairs $(i, j) \in \{1, \dots, s\} \times \{1, \dots, t\}$ such that

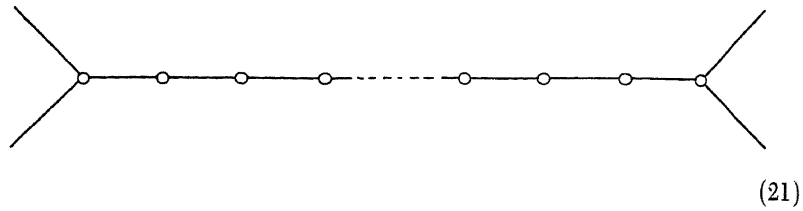
$$v_i = w_j \in W, \quad f_i = g_{j+1}, \quad \text{and} \quad f_{i+1} \neq g_j. \quad (19)$$

Note that the right-hand side of (12) is equal to $\lambda + \mu + \nu$. To see that $\text{mincr}(C, D) \leq \lambda + \mu + \nu$, note that μ is equal to the number of pairs $(i, j) \in$

$\{1, \dots, s\} \times \{1, \dots, t\}$ such that there exists a number $b \geq 1$ with

$$\begin{aligned} f_i &\neq g_j, & v_i &= w_j, & f_{i+1} &= g_{j+1}, & v_{i+1} &= w_{j+1}, \\ f_{i+2} &= g_{j+2}, \dots, & v_{i+b} &= w_{j+b}, & f_{i+b+1} &\neq g_{j+b+1}, \end{aligned} \quad (20)$$

which corresponds to pictures of type



Similarly, ν is equal to the number of pairs $(i, j) \in \{1, \dots, s\} \times \{1, \dots, t\}$ such that there exists a number $b \geq 1$ with

$$\begin{aligned} f_i &\neq g_{j+1}, & v_i &= w_j, & f_{i+1} &= g_j, & v_{i+1} &= w_{j-1}, \\ f_{i+2} &= g_{j-1}, \dots, & v_{i+b} &= w_{j-b}, & f_{i+b+1} &\neq g_{j-b}. \end{aligned} \quad (22)$$

Again, this corresponds to a picture of type (21).

Since each of the intersections of type (21) can be replaced by parts that have one crossing or none at all, we obtain $\text{mincr}(C, D) \leq \lambda + \mu + \nu$. ■

Next we study the pattern of the C_{ij} at one fixed vertex v of degree 4. Again, let the neighborhood of v be as in (9), and denote for $g, h \in \{1, 2, 3, 4\}$

$$\alpha_{gh}^v := \alpha_{gh} := \sum_{i=1}^k \sum_{j=1}^{t_i} \lambda_{ij} \alpha_{gh}^v(C_{ij}). \quad (23)$$

Then:

LEMMA B. *For each fixed vertex v of degree 4,*

$$\begin{aligned} 2\alpha_{13}\alpha_{24} + \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{14} + \alpha_{12}\alpha_{23} + \alpha_{12}\alpha_{24} + \alpha_{13}\alpha_{14} + \alpha_{13}\alpha_{23} + \alpha_{13}\alpha_{34} \\ + \alpha_{14}\alpha_{24} + \alpha_{14}\alpha_{34} + \alpha_{23}\alpha_{24} + \alpha_{23}\alpha_{34} + \alpha_{24}\alpha_{34} \leq 2, \end{aligned} \quad (24)$$

with equality only if $\alpha_{13} = \alpha_{24} = 1$ and $\alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{14} = 0$.

Proof of Lemma B. The left-hand side of (24) is not larger than the first expression in the following series of inequalities (as the latter is obtained by adding $2\alpha_{12}\alpha_{34} + 2\alpha_{14}\alpha_{23}$):

$$\begin{aligned}
 & 2\alpha_{13}\alpha_{24} + 2\alpha_{12}\alpha_{34} + 2\alpha_{14}\alpha_{23} + \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{14} + \alpha_{12}\alpha_{23} + \alpha_{12}\alpha_{24} + \alpha_{13}\alpha_{14} \\
 & \quad + \alpha_{13}\alpha_{23} + \alpha_{13}\alpha_{34} + \alpha_{14}\alpha_{24} + \alpha_{14}\alpha_{34} + \alpha_{23}\alpha_{24} + \alpha_{23}\alpha_{34} + \alpha_{24}\alpha_{34} \\
 = & \frac{1}{2} [\alpha_{12}(\alpha_{13} + \alpha_{14} + \alpha_{23} + \alpha_{24} + 2\alpha_{34}) + \alpha_{13}(\alpha_{12} + \alpha_{14} + \alpha_{23} + \alpha_{34} + 2\alpha_{24}) \\
 & \quad + \alpha_{14}(\alpha_{12} + \alpha_{13} + \alpha_{24} + \alpha_{34} + 2\alpha_{23}) + \alpha_{23}(\alpha_{12} + \alpha_{24} + \alpha_{13} + \alpha_{34} + 2\alpha_{14}) \\
 & \quad + \alpha_{24}(\alpha_{12} + \alpha_{23} + \alpha_{14} + \alpha_{34} + 2\alpha_{13}) + \alpha_{34}(\alpha_{13} + \alpha_{23} + \alpha_{14} + \alpha_{24} + 2\alpha_{12})] \\
 = & \frac{1}{2} [\alpha_{12}(\delta_3 + \delta_4) + \alpha_{13}(\delta_2 + \delta_4) + \alpha_{14}(\delta_2 + \delta_3) + \alpha_{23}(\delta_1 + \delta_4) \\
 & \quad + \alpha_{24}(\delta_1 + \delta_3) + \alpha_{34}(\delta_1 + \delta_2)], \quad (25)
 \end{aligned}$$

where, for $g \in \{1, 2, 3, 4\}$,

$$\delta_g := \sum_{i=1}^k \sum_{j=1}^{l_i} \lambda_{ij} \chi^{C_{ij}}(e_g). \quad (26)$$

So by (6c), $\delta_g \leq 1$ for each $g \in \{1, 2, 3, 4\}$. Moreover,

$$\begin{aligned}
 \delta_1 &= \alpha_{12} + \alpha_{13} + \alpha_{14}, & \delta_2 &= \alpha_{12} + \alpha_{23} + \alpha_{24}, \\
 \delta_3 &= \alpha_{13} + \alpha_{23} + \alpha_{34}, & \delta_4 &= \alpha_{14} + \alpha_{24} + \alpha_{34}.
 \end{aligned} \quad (27)$$

Hence the last expression in (25) is not larger than the first expression in

$$\alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{23} + \alpha_{24} + \alpha_{34} = \frac{1}{2}(\delta_1 + \delta_2 + \delta_3 + \delta_4) \leq 2. \quad (28)$$

This proves the inequality (24). In order to have equality we should have

$$\alpha_{12}\alpha_{34} = 0, \quad \alpha_{14}\alpha_{23} = 0, \quad \delta_1 = \delta_2 = \delta_3 = \delta_4 = 1. \quad (29)$$

Now (27) and (29) imply

$$\begin{aligned}\alpha_{12} &= \frac{1}{2}(\delta_1 + \delta_2 - \delta_3 - \delta_4) + \alpha_{34} = \alpha_{34}, \\ \alpha_{14} &= \frac{1}{2}(\delta_1 + \delta_4 - \delta_2 - \delta_3) + \alpha_{23} = \alpha_{23}.\end{aligned}\tag{30}$$

Hence $\alpha_{12} = \alpha_{34} = \alpha_{14} = \alpha_{23} = 0$ and $\alpha_{13} = \alpha_{24} = 1$. \blacksquare

From Lemmas A and B we derive

$$\begin{aligned}& \sum_{i=1}^k \sum_{i'=1}^k \text{mincr}(C_i, C_{i'}) \\ &= \sum_{i=1}^k \sum_{j=1}^{t_i} \sum_{i'=1}^k \sum_{j'=1}^{t_{i'}} \lambda_{ij} \lambda_{i'j'} \text{mincr}(C_{ij}, C_{i'j'}) \\ &\leq \sum_{i=1}^k \sum_{j=1}^{t_i} \sum_{i'=1}^k \sum_{j'=1}^{t_{i'}} \lambda_{ij} \lambda_{i'j'} \sum_{v \in W} (\alpha_{13}^v(C_{ij}) \alpha_{24}^v(C_{i'j'}) + \alpha_{24}^v(C_{ij}) \alpha_{13}^v(C_{i'j'})) \\ &\quad + \frac{1}{2} \left\{ [\alpha_{12}^v(C_{ij}) + \alpha_{34}^v(C_{ij})] [\alpha_{13}^v(C_{i'j'}) + \alpha_{14}^v(C_{i'j'}) + \alpha_{23}^v(C_{i'j'}) + \alpha_{24}^v(C_{i'j'})] \right. \\ &\quad + [\alpha_{13}^v(C_{ij}) + \alpha_{24}^v(C_{ij})] [\alpha_{12}^v(C_{i'j'}) + \alpha_{14}^v(C_{i'j'}) + \alpha_{23}^v(C_{i'j'}) + \alpha_{34}^v(C_{i'j'})] \\ &\quad \left. + [\alpha_{14}^v(C_{ij}) + \alpha_{23}^v(C_{ij})] [\alpha_{12}^v(C_{i'j'}) + \alpha_{13}^v(C_{i'j'}) + \alpha_{24}^v(C_{i'j'}) + \alpha_{34}^v(C_{i'j'})] \right\} \\ &= \sum_{v \in W} \left\{ \alpha_{13}^v \alpha_{24}^v + \alpha_{24}^v \alpha_{13}^v + \frac{1}{2} [(\alpha_{12}^v + \alpha_{34}^v)(\alpha_{13}^v + \alpha_{14}^v + \alpha_{23}^v + \alpha_{24}^v) \right. \\ &\quad \left. + (\alpha_{13}^v + \alpha_{24}^v)(\alpha_{12}^v + \alpha_{14}^v + \alpha_{23}^v + \alpha_{34}^v) + (\alpha_{14}^v + \alpha_{23}^v)(\alpha_{12}^v + \alpha_{13}^v + \alpha_{24}^v + \alpha_{34}^v) \right\} \\ &= \sum_{v \in W} (2\alpha_{13}^v \alpha_{24}^v + \alpha_{12}^v \alpha_{13}^v + \alpha_{12}^v \alpha_{14}^v + \alpha_{12}^v \alpha_{23}^v + \alpha_{12}^v \alpha_{24}^v + \alpha_{13}^v \alpha_{14}^v + \alpha_{13}^v \alpha_{23}^v + \alpha_{13}^v \alpha_{34}^v \\ &\quad + \alpha_{14}^v \alpha_{24}^v + \alpha_{14}^v \alpha_{34}^v + \alpha_{23}^v \alpha_{24}^v + \alpha_{23}^v \alpha_{34}^v + \alpha_{24}^v \alpha_{34}^v + \alpha_{24}^v \alpha_{34}^v) \\ &\leq 2|W| = \sum_{i=1}^{k'} \sum_{i'=1}^{k'} \text{mincr}(C_i', C_{i'}').\end{aligned}\tag{31}$$

By our assumption (3), we should have equality throughout in (31). Hence by Lemma B, for each vertex v in W ,

$$\alpha_{13}^v = \alpha_{24}^v = 1, \quad \alpha_{12}^v = \alpha_{23}^v = \alpha_{34}^v = \alpha_{14}^v = 0. \tag{32}$$

So (10) holds, and hence we have (i) in our theorem. □□

Our theorem above can be formulated equivalently as the nonsingularity of a certain infinite symmetric matrix. Let \mathcal{C} be the family of equivalence classes of closed curves on S , with respect to the equivalence relation \sim . For $\Gamma, \Delta \in \mathcal{C}$ we define

$$\text{mincr}(\Gamma, \Delta) := \text{mincr}(C, D) \tag{33}$$

for (arbitrary) $C \in \Gamma$ and $D \in \Delta$. So mincr is considered also as a function from $\mathcal{C} \times \mathcal{C}$ to \mathbb{Z}_+ (= set of nonnegative integers). We can represent this function as an infinite symmetric matrix M , with both rows and columns indexed by \mathcal{C} .

The rows of M are not linearly independent. First of all, the row corresponding to the trivial class $\langle 0 \rangle$ is all-zero (where 0 is a homotopically trivial closed curve, and where $\langle \dots \rangle$ denotes the equivalence class containing \dots). Moreover, the rows corresponding to $\langle C \rangle$ and to $\langle C^{-1} \rangle$ are the same, since $\text{mincr}(C^{-1}, D) = \text{mincr}(C, D)$ for each closed curve D . More generally, it is shown in [2] that for each pair of closed curves C, D on S and each $n \in \mathbb{Z}$,

$$\text{mincr}(C^n, D) = |n| \text{mincr}(C, D). \tag{34}$$

So the row corresponding to $\langle C^n \rangle$ is a multiple of the row corresponding to $\langle C \rangle$.

Now the theorem above actually says that (34) generate *all* linear dependencies of rows of M . To explain this, we mention the following result of [2]:

$$\text{for each homotopically nontrivial closed curve } C \text{ on } S \text{ there exists a primitive closed curve } D \text{ on } S \text{ and an integer } n \geq 1 \text{ such that } C \sim D^n. \text{ The integer } n \text{ and closed curve } D \text{ are unique (up to homotopy).} \tag{35}$$

Let $\mathcal{C}_p \subseteq \{\langle C \rangle \mid C \text{ a primitive closed curve}\}$, so that for each primitive closed curve C exactly one of $\langle C \rangle$ and $\langle C^{-1} \rangle$ belongs to \mathcal{C}_p , which we denote by $[C]$. Let M' be the $\mathcal{C}_p \times \mathcal{C}_p$ submatrix of M . Then the following is equivalent to our theorem above:

EQUIVALENT FORM OF THE THEOREM. *The matrix M' is nonsingular, i.e., the rows of M' are linearly independent.*

Proof of Equivalence.

I. We first derive the equivalent form from the theorem. Suppose M' has linearly dependent rows. That is, there are distinct $\Gamma_1, \dots, \Gamma_t \in \mathcal{C}_p$ and $\lambda_1, \dots, \lambda_t \in \mathbb{R} \setminus \{0\}$ (with $t \geq 1$) such that for each $\Delta \in \mathcal{C}_p$

$$\sum_{i=1}^t \lambda_i \operatorname{mincr}(\Gamma_i, \Delta) = 0. \quad (36)$$

Since M' is an integer matrix, we may assume that the λ_i are rational, and hence integer. By repeating each Γ_i $|\lambda_i|$ times, we obtain $\Gamma'_1, \dots, \Gamma'_{t'}$ and $\Gamma''_1, \dots, \Gamma''_{t''}$ in \mathcal{C}_p (with $t' + t'' \geq 1$), so that for each $\Delta \in \mathcal{C}_p$

$$\sum_{i=1}^{t'} \operatorname{mincr}(\Gamma'_i, \Delta) = \sum_{i=1}^{t''} \operatorname{mincr}(\Gamma''_i, \Delta), \quad (37)$$

and so that $\{\Gamma'_1, \dots, \Gamma'_{t'}\} \cap \{\Gamma''_1, \dots, \Gamma''_{t''}\} = \emptyset$. Now (34) and (35) imply that (37) in fact holds for every $\Delta \in \mathcal{C}$. But then our theorem gives that $t' = t''$, and there exists a permutation π of $\{1, \dots, t'\}$ such that $\Gamma'_i = \Gamma''_{\pi(i)}$ for each $i = 1, \dots, t'$. This is a contradiction.

II. To see the reverse implication, note that condition (ii) of the Theorem implies that for each $\Delta \in \mathcal{C}_p$

$$\sum_{i=1}^k \operatorname{mincr}([C_i], \Delta) = \sum_{i=1}^{k'} \operatorname{mincr}([C'_i], \Delta). \quad (38)$$

Since by the equivalent form of the theorem the rows of M' are linearly

independent, we must have $k = k'$ and $[C_i] = [C'_{\pi(i)}]$ for each $i = 1, \dots, k$, for some permutation π of $\{1, \dots, k\}$. ■

REFERENCES

- 1 R. Baer, Kurventypen auf Flächen, *J. Reine Angew. Math.* 156:231–246 (1927).
- 2 A. Schrijver, Decomposition of Graphs on Surfaces and a Homotopic Circulation Theorem, Report OS-R8719, Mathematical Centre, Amsterdam, 1987.

Received 4 February 1988; final manuscript accepted 6 May 1988